ON A GENERAL THEORY OF ANISOTROPIC SHELLS

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1. We consider a thin anisotropic shell of constant thickness h. Assume that the material of the shell obeys the generalized Hooke's law and that at each point there is only one plane of elastic symmetry, parallel to the middle surface of the shell. The latter surface will be used as surface of coordinates, and the shell will be referred to curvilinear orthogonal coordinates a and β, which coincide with the principal curvature lines of that surface. Let γ represent the distance, measured along the normal, between the point (a, β) of the middle surface and the point (a, β, γ) of the shell. We assume that

(a) the line elements of the shell, normal to the middle surface, do not change their lengths after deformation;

(b) the normal stresses* σ_γ are small as compared with the stresses σ_a, σ_β and $\tau_{a\beta}$;

(c) the shear stresses $\tau_{a\gamma}$ and $\tau_{\beta\gamma}$ vary in the direction of the thickness of the shell in accordance with the law of the quadratic parabola [13].

Being more rigorous in the formulation of the hypotheses [2,5], we can state here the assumptions (a) and (b) in the following form:

(a) $e_{\gamma\gamma} = 0$ approximately;

(b) the stresses σ_γ do not exert any essential influence on the strain components e_{aa} and $e_{\beta\beta}$ and they can be neglected in the corresponding equations of the generalized Hooke's law.

2. By virtue of the assumption (c) concerning the shear stresses $\tau_{a\gamma}$ and $\tau_{\beta\gamma}$ we have

$$\begin{aligned}\tau_{\alpha\gamma} &= \frac{X^+ - X^-}{2} + \frac{\gamma}{h}(X^+ + X^-) + \frac{1}{2}\left(\gamma^2 - \frac{h^2}{4}\right)\varphi(\alpha, \beta)\\ \tau_{\beta\gamma} &= \frac{Y^+ - Y^-}{2} + \frac{\gamma}{h}(Y^+ + Y^-) + \frac{1}{2}\left(\gamma^2 - \frac{h^2}{4}\right)\psi(\alpha, \beta)\end{aligned} \tag{2.1}$$

* Here and in the following we adopt the well-known notations used in the theory of shells.

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where $X^+(\alpha, \beta)$, $Y^+(\alpha, \beta)$ and $X^-(\alpha, \beta)$, $Y^-(\alpha, \beta)$ are the components along the axes of the moving trihedron (in the directions of the positive tangents to the lines β = const., α = const., respectively) of the intensity vectors of the surface loads, applied to the boundary surfaces $\gamma = \frac{1}{2}h$ and $\gamma = -\frac{1}{2}h$, respectively, while $\phi(\alpha, \beta)$, $\psi(\alpha, \beta)$ are unknown functions. Substituting the values of the tangential stresses $\tau_{\alpha\gamma}$ and $\tau_{\beta\gamma}$ from (2.1) into the corresponding equations of the generalized Hooke's law [6], we obtain for the shear strain components $e_{\alpha\gamma}$ and $e_{\beta\gamma}$ the formulas

$$
\begin{aligned}
e_{\alpha\gamma} &= X + \frac{\gamma}{h} X' + \frac{1}{2}\left(\gamma^2 - \frac{h^2}{4}\right)\Phi_1(\alpha, \beta) \\
e_{\beta\gamma} &= Y + \frac{\gamma}{h} Y' + \frac{1}{2}\left(\gamma^2 - \frac{h^2}{4}\right)\Phi_2(\alpha, \beta)
\end{aligned}
\tag{2.2}
$$

Here we have introduced the following notations:

$$
\begin{aligned}
X &= \frac{1}{2}[a_{55}(X^+ - X^-) + a_{45}(Y^+ - Y^-)] \\
Y &= \frac{1}{2}[a_{44}(Y^+ - Y^-) + a_{45}(X^+ - X^-)]
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
X' &= a_{55}(X^+ + X^-) + a_{45}(Y^+ + Y^-) \\
Y' &= a_{44}(Y^+ + Y^-) + a_{45}(X^+ + X^-)
\end{aligned}
\tag{2.4}
$$

$$
\Phi_1 = a_{55}\varphi + a_{45}\psi, \qquad \Phi_2 = a_{44}\psi + a_{45}\varphi \tag{2.5}
$$

where the quantities a_{ik} are elastic constants [6].

From the equations of the three-dimensional theory of elasticity we have for the strain components [1]

$$
\begin{aligned}
e_{\alpha\alpha} &= \frac{1}{H_1}\frac{\partial u_\alpha}{\partial \alpha} + \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \beta} u_\beta + \frac{1}{H_1}\frac{\partial H_1}{\partial \gamma} u_\gamma \\
e_{\beta\beta} &= \frac{1}{H_2}\frac{\partial u_\beta}{\partial \beta} + \frac{1}{H_2}\frac{\partial H_2}{\partial \gamma} u_\gamma + \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha} u_\alpha
\end{aligned}
\tag{2.6}
$$

$$
e_{\gamma\gamma} = \frac{\partial u_\gamma}{\partial \gamma} \tag{2.7}
$$

$$
e_{\alpha\beta} = \frac{H_1}{H_2}\frac{\partial}{\partial \beta}\left(\frac{1}{H_1} u_\alpha\right) + \frac{H_2}{H_1}\frac{\partial}{\partial \alpha}\left(\frac{1}{H_2} u_\beta\right) \tag{2.8}
$$

$$
\begin{aligned}
e_{\beta\gamma} &= H_2 \frac{\partial}{\partial \gamma}\left(\frac{1}{H_2} u_\beta\right) + \frac{1}{H_2}\frac{\partial}{\partial \beta} u_\gamma \\
e_{\gamma\alpha} &= \frac{1}{H_1}\frac{\partial}{\partial \alpha} u_\gamma + H_1 \frac{\partial}{\partial \gamma}\left(\frac{1}{H_1} u_\alpha\right)
\end{aligned}
\tag{2.9}
$$

$$
H_1 = A(1 + k_1\gamma), \qquad H_2 = B(1 + k_2\gamma) \tag{2.10}
$$

In these formulas $A = A(\alpha, \beta)$ and $B = B(\alpha, \beta)$ are the coefficients of the first quadratic form of the middle surface, $k_1 = k_1(\alpha, \beta)$ and $k_2 = k_2(\alpha, \beta)$ are the principal curvatures of the middle surface, $u_\alpha = u_\alpha(\alpha, \beta, \gamma)$, $u_\beta = u_\beta(\alpha, \beta, \gamma)$ and $u_\gamma = u_\gamma(\alpha, \beta, \gamma)$ are the displacement components of an arbitrary point of the shell in the directions of the tangents to the coordinate lines, respectively.

On the basis of the assumption (a) we find from (2.7)

$$\frac{\partial u_\gamma}{\partial \gamma} = 0, \quad u_\gamma = u_\gamma(\alpha, \beta) = w(\alpha, \beta) \tag{2.11}$$

Thus, like in all existing theories of thin shells, the displacement u_γ of any point of the shell is independent of the coordinate γ. This displacement component has for all points of a line element of a normal to the shell a constant value, equal to the normal displacement component $w = w(\alpha, \beta)$ of the corresponding point of the middle surface of the shell.

Substituting the expressions for $e_{\alpha\gamma}$, $e_{\beta\gamma}$, H_1, H_2 and u_γ from (2.2), (2.10) and (2.11) into equations (2.9), we obtain differential equations for the displacement components u_α and u_β. Integrating these equations and taking into consideration that $u_\alpha = u(\alpha, \beta)$ and $u_\beta = v(\alpha, \beta)$ when $\gamma = 0$, we find

$$\begin{gathered} u_\alpha = (1 + k_1\gamma)\, u - \frac{\gamma}{A}\frac{\partial w}{\partial \alpha} - \gamma\left(1 + \gamma\frac{k_1}{2}\right)\frac{h^2}{8}\Phi_1 + \\ + \gamma^3\left(1 + \gamma\frac{k_1}{4}\right)\frac{1}{6}\Phi_1 + \gamma\left(1 + \gamma\frac{k_1}{2}\right)X + \gamma^2\left(1 + \gamma\frac{k_1}{3}\right)\frac{1}{2h}X' \\ u_\beta = (1 + k_2\gamma)\, v - \frac{\gamma}{B}\frac{\partial w}{\partial \beta} - \gamma\left(1 + \gamma\frac{k_2}{2}\right)\frac{h^2}{8}\Phi_2 + \\ + \gamma^3\left(1 + \gamma\frac{k_2}{4}\right)\frac{1}{6}\Phi_2 + \gamma\left(1 + \gamma\frac{k_2}{2}\right)Y + \gamma^2\left(1 + \gamma\frac{k_2}{3}\right)\frac{1}{2h}Y' \end{gathered} \tag{2.12}$$

where $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ are the tangential displacement components of the corresponding point of the middle surface.

In the process of deriving the formulas (2.12) the accuracy was being confined to consideration of quantities up to those of the order of magnitude of γk_i, i.e. whenever a sufficiently precise estimation was possible, terms of the order of magnitude of $(\gamma k_i)^2$ were being neglected in comparison with unity.

Our formulas (2.12) show that, in contrast to known theories of thin shells [1,2,5,7], the tangential displacement components u_α and u_β of any point of the shell at a distance γ from the middle surface are, in the case considered here, as in the publications [8,9], non-linear functions of the distance γ.

By virtue of (2.12) the strain components $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$ can be expressed by polynomials in powers of γ, namely

$$e_{\alpha\alpha} = \varepsilon_1 + \gamma \varkappa_1 + \gamma^2 \eta_1 + \gamma^3 \theta_1 + \gamma^4 \xi_1$$
$$e_{\beta\beta} = \varepsilon_2 + \gamma \varkappa_2 + \gamma^2 \eta_2 + \gamma^3 \theta_2 + \gamma^4 \xi_2 \quad (2.13)$$
$$e_{\alpha\beta} = \omega + \gamma \tau + \gamma^2 \nu + \gamma^3 \lambda + \gamma^4 \zeta$$

Substituting the values of u_α, u_β, u_γ from (2.12) and (2.11), respectively, into the relations (2.6) and (2.8), and comparing the resulting expressions for the strain components $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$ with the corresponding expressions (2.13), we obtain the following formulas for the coefficients of the expansions:

$$\varepsilon_1 = \varepsilon_1^\circ = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} v + k_1 w \quad (2.14)$$

$$\varepsilon_2 = \varepsilon_2^\circ = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} u + k_2 w \quad (2.15)$$

$$\omega = \omega^\circ = \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) \quad (2.16)$$

$$\varkappa_1 = \varkappa_1^\circ - \frac{h^2}{8}\left(\frac{1}{A}\frac{\partial \Phi_1}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\Phi_2\right) + \frac{1}{A}\frac{\partial X}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta} Y \quad (2.17)$$

$$\varkappa_2 = \varkappa_2^\circ - \frac{h^2}{8}\left(\frac{1}{B}\frac{\partial \Phi_2}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\Phi_1\right) + \frac{1}{B}\frac{\partial Y}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha} X \quad (2.18)$$

$$\tau = \tau^\circ - \frac{h^2}{8}\left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{\Phi_1}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{\Phi_2}{B}\right)\right] + \left[\frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{X}{A}\right) + \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{Y}{B}\right)\right] \quad (2.19)$$

$$\begin{aligned} \eta_1 = & -k_1 \frac{1}{A}\frac{\partial k_1}{\partial \alpha} u + k_1 \frac{1}{A}\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial w}{\partial \alpha}\right) + \\ & + k_1 \frac{1}{AB^2}\frac{\partial A}{\partial \beta}\frac{\partial w}{\partial \beta} + \frac{h^2}{16}\left(k_1 \frac{1}{A}\frac{\partial \Phi_1}{\partial \alpha} - \frac{1}{A}\frac{\partial k_1}{\partial \alpha}\Phi_1\right) + \\ & + \frac{h^2}{8}\left(k_1 - \frac{1}{2}k_2\right)\frac{1}{AB}\frac{\partial A}{\partial \beta}\Phi_2 - \frac{1}{2}\left(k_1 \frac{1}{A}\frac{\partial X}{\partial \alpha} - \frac{1}{A}\frac{\partial k_1}{\partial \alpha} X\right) - \\ & - \left(k_1 - \frac{1}{2}k_2\right)\frac{1}{AB}\frac{\partial A}{\partial \beta} Y + \frac{1}{2h}\frac{1}{A}\frac{\partial X'}{\partial \alpha} + \frac{1}{2h}\frac{1}{AB}\frac{\partial A}{\partial \beta} Y' \end{aligned} \quad (2.20)$$

$$\begin{aligned} \eta_2 = & -k_2 \frac{1}{B}\frac{\partial k_2}{\partial \beta} v + k_2 \frac{1}{B}\frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial w}{\partial \beta}\right) + \\ & + k_2 \frac{1}{BA^2}\frac{\partial B}{\partial \alpha}\frac{\partial w}{\partial \alpha} + \frac{h^2}{16}\left(k_2 \frac{1}{B}\frac{\partial \Phi_2}{\partial \beta} - \frac{1}{B}\frac{\partial k_2}{\partial \beta}\Phi_2\right) + \\ & + \frac{h^2}{8}\left(k_2 - \frac{1}{2}k_1\right)\frac{1}{AB}\frac{\partial B}{\partial \alpha}\Phi_1 - \frac{1}{2}\left(k_2 \frac{1}{B}\frac{\partial Y}{\partial \beta} - \frac{1}{B}\frac{\partial k_2}{\partial \beta} Y\right) - \\ & - \left(k_2 - \frac{1}{2}k_1\right)\frac{1}{AB}\frac{\partial B}{\partial \alpha} X + \frac{1}{2h}\frac{1}{B}\frac{\partial Y'}{\partial \beta} + \frac{1}{2h}\frac{1}{AB}\frac{\partial B}{\partial \alpha} X' \end{aligned} \quad (2.21)$$

$$\begin{aligned}\nu = {} & k_2 \frac{1}{B} \frac{\partial}{\partial\beta}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) + k_1 \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \\ & - k_2 \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - k_1 \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \\ & + \frac{h^2}{16}\left[\frac{1}{B}(2k_2 - k_1)\frac{\partial\Phi_1}{\partial\beta} - \frac{1}{B}\left(\frac{\partial k_1}{\partial\beta} + \frac{1}{A}\frac{\partial A}{\partial\beta}k_1\right)\Phi_1\right] + \\ & + \frac{h^2}{16}\left[\frac{1}{A}(2k_1 - k_2)\frac{\partial\Phi_2}{\partial\alpha} - \frac{1}{A}\left(\frac{\partial k_2}{\partial\alpha} + \frac{1}{B}\frac{\partial B}{\partial\alpha}k_2\right)\Phi_2\right] + \\ & + \frac{1}{2B}\left[\frac{\partial}{\partial\beta}(k_1X) - 2k_2\frac{\partial X}{\partial\beta} + \frac{1}{A}\frac{\partial A}{\partial\beta}k_1X\right] + \\ & + \frac{1}{2A}\left[\frac{\partial}{\partial\alpha}(k_2Y) - 2k_1\frac{\partial Y}{\partial\alpha} + \frac{1}{B}\frac{\partial B}{\partial\alpha}k_2Y\right] + \\ & + \frac{1}{2h}\left[\frac{1}{B}\left(\frac{\partial X'}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}X'\right) + \frac{1}{A}\left(\frac{\partial Y'}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}Y'\right)\right]\end{aligned} \qquad (2.22)$$

$$\begin{aligned}\theta_1 = {} & \frac{1}{6}\frac{1}{A}\frac{\partial\Phi_1}{\partial\alpha} + \frac{1}{6}\frac{1}{AB}\frac{\partial A}{\partial\beta}\Phi_2 + \frac{h^2}{16}k_1\frac{1}{A}\frac{\partial k_1}{\partial\alpha}\Phi_1 - \\ & - \frac{1}{2}k_1\frac{1}{A}\frac{\partial k_1}{\partial\alpha}X - \frac{1}{3h}\left(k_1\frac{1}{A}\frac{\partial X'}{\partial\alpha} - \frac{1}{2}\frac{1}{A}\frac{\partial k_1}{\partial\alpha}X'\right) - \\ & - \frac{1}{2h}\left(k_1 - \frac{1}{3}k_2\right)\frac{1}{AB}\frac{\partial A}{\partial\beta}Y'\end{aligned} \qquad (2.23)$$

$$\begin{aligned}\theta_2 = {} & \frac{1}{6}\frac{1}{B}\frac{\partial\Phi_2}{\partial\beta} + \frac{1}{6}\frac{1}{AB}\frac{\partial B}{\partial\alpha}\Phi_1 + \frac{h^2}{16}k_2\frac{1}{B}\frac{\partial k_2}{\partial\beta}\Phi_2 - \frac{1}{2}k_2\frac{1}{B}\frac{\partial k_2}{\partial\beta}Y - \\ & - \frac{1}{3h}\left(k_2\frac{1}{B}\frac{\partial Y'}{\partial\beta} - \frac{1}{2}\frac{1}{B}\frac{\partial k_2}{\partial\beta}Y'\right) - \frac{1}{2h}\left(k_2 - \frac{1}{3}k_1\right)\frac{1}{AB}\frac{\partial B}{\partial\alpha}X'\end{aligned} \qquad (2.24)$$

$$\begin{aligned}\lambda = {} & \frac{1}{6}\frac{1}{B}\frac{\partial\Phi_1}{\partial\beta} - \frac{1}{6}\frac{1}{AB}\frac{\partial A}{\partial\beta}\Phi_1 + \frac{1}{6}\frac{1}{A}\frac{\partial\Phi_2}{\partial\alpha} - \\ & - \frac{1}{6}\frac{1}{AB}\frac{\partial B}{\partial\alpha}\Phi_2 + \frac{1}{h}\left[\frac{1}{6}\frac{1}{B}\frac{\partial}{\partial\beta}(k_1X') - \frac{1}{2}k_2\frac{1}{B}\frac{\partial X'}{\partial\beta} + \right. \\ & \left. + \frac{1}{3}k_1\frac{1}{AB}\frac{\partial A}{\partial\beta}X'\right] + \frac{1}{h}\left[\frac{1}{A}\frac{1}{6}\frac{\partial}{\partial\alpha}(k_2Y') - \right. \\ & \left. - \frac{1}{2}k_1\frac{1}{A}\frac{\partial Y'}{\partial\alpha} + \frac{1}{3}k_2\frac{1}{AB}\frac{\partial B}{\partial\alpha}Y'\right]\end{aligned} \qquad (2.25)$$

$$\xi_1 = \frac{1}{24}\frac{1}{A}\frac{\partial k_1}{\partial\alpha}\Phi_1 + \frac{1}{6}\frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{k_2}{4} - k_1\right)\Phi_2 - \frac{1}{8}\frac{1}{A}k_1\frac{\partial\Phi_1}{\partial\alpha} \qquad (2.26)$$

$$\xi_2 = \frac{1}{24}\frac{1}{B}\frac{\partial k_2}{\partial\beta}\Phi_2 + \frac{1}{6}\frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{k_1}{4} - k_2\right)\Phi_1 - \frac{1}{8}\frac{1}{B}k_2\frac{\partial\Phi_2}{\partial\beta} \qquad (2.27)$$

$$\begin{aligned}\zeta = {} & \frac{1}{B}\left[\frac{1}{24}\frac{\partial}{\partial\beta}(k_1\Phi_1) - \frac{1}{6}k_2\frac{\partial\Phi_1}{\partial\beta} + \frac{1}{8}\frac{1}{A}\frac{\partial A}{\partial\beta}k_1\Phi_1\right] + \\ & + \frac{1}{A}\left[\frac{1}{24}\frac{\partial}{\partial\alpha}(k_2\Phi_2) - \frac{1}{6}k_1\frac{\partial\Phi_2}{\partial\alpha} + \frac{1}{8}\frac{1}{B}\frac{\partial B}{\partial\alpha}k_2\Phi_2\right]\end{aligned} \qquad (2.28)$$

In formulas (2.17) to (2.19) we have, in conformity with the usual definition of curvature changes and torsion of the middle surface of the shell [2,5],

$$\varkappa_1^\circ = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} - \frac{u}{R_1}\right) - \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} - \frac{v}{R_2}\right) \tag{2.29}$$

$$\varkappa_2^\circ = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta} - \frac{v}{R_2}\right) - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha} - \frac{u}{R_1}\right) \tag{2.30}$$

$$\begin{aligned}\tau^\circ = &-\frac{2}{AB}\left(\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}\right) + \\ &+\frac{2}{R_1}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{1}{AB}\frac{\partial A}{\partial\beta}u\right) + \frac{2}{R_2}\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}v\right)\end{aligned} \tag{2.31}$$

where $R_1 = R_1(\alpha, \beta)$ and $R_2 = R_2(\alpha, \beta)$ are the principal radii of curvature of the middle surface.

Considering the expansions (2.13) we note that they have some similarity with the analogous expansions used in ref. [1]; the similarity is, however, only a superficial one. In the determination of the strain components $e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\alpha\beta}$ ref. [1] actually uses expressions in terms of powers of γ keeping at the same time the hypothesis of non-deformable normals [1,2], while in the present paper, as in the publications [8,9], the relations (2.13) are being obtained on the basis of the basic assumptions of the theory offered here.

On the basis of (2.13) and of the original assumption (b) we derive from the generalized Hooke's law the following expressions for the stress components $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$, $\tau_{\alpha\beta}$:

$$\begin{aligned}\sigma_{\alpha\alpha} = &B_{11}\varepsilon_1 + B_{12}\varepsilon_2 + B_{16}\omega + \gamma(B_{11}\varkappa_1 + B_{12}\varkappa_2 + \\ &+ B_{16}\tau) + \gamma^2(B_{11}\eta_1 + B_{12}\eta_2 + B_{16}\nu) + \gamma^3(B_{11}\theta_1 + \\ &+ B_{12}\theta_2 + B_{16}\lambda) + \gamma^4(B_{11}\xi_1 + B_{12}\xi_2 + B_{16}\zeta)\end{aligned} \tag{2.32}$$

$$\begin{aligned}\sigma_{\beta\beta} = &B_{22}\varepsilon_2 + B_{12}\varepsilon_1 + B_{26}\omega + \gamma(B_{22}\varkappa_2 + B_{12}\varkappa_1 + \\ &+ B_{26}\tau) + \gamma^2(B_{22}\eta_2 + B_{12}\eta_1 + B_{26}\nu) + \gamma^3(B_{22}\theta_2 + \\ &+ B_{12}\theta_1 + B_{26}\lambda) + \gamma^4(B_{22}\xi_2 + B_{12}\xi_1 + B_{26}\zeta)\end{aligned} \tag{2.33}$$

$$\begin{aligned}\tau_{\alpha\beta} = &B_{16}\varepsilon_1 + B_{26}\varepsilon_2 + B_{66}\omega + \gamma(B_{16}\varkappa_1 + B_{26}\varkappa_2 + B_{66}\tau) + \\ &+ \gamma^2(B_{16}\eta_1 + B_{26}\eta_2 + B_{66}\nu) + \gamma^3(B_{16}\theta_1 + B_{26}\theta_2 + B_{66}\lambda) + \\ &+ \gamma^4(B_{16}\xi_1 + B_{26}\xi_2 + B_{66}\zeta)\end{aligned} \tag{2.34}$$

In these formulas the constants B_{ik} are given by the following expressions in terms of the elastic constants a_{ik} [10,11]:

$$\begin{aligned}&B_{11} = \frac{a_{22}a_{66} - a_{26}^2}{\Omega}, \quad B_{12} = \frac{a_{16}a_{26} - a_{12}a_{66}}{\Omega}, \quad B_{16} = \frac{a_{12}a_{26} - a_{22}a_{16}}{\Omega} \\ &B_{22} = \frac{a_{11}a_{66} - a_{16}^2}{\Omega}, \quad B_{66} = \frac{a_{11}a_{22} - a_{12}^2}{\Omega}, \quad B_{26} = \frac{a_{12}a_{16} - a_{11}a_{26}}{\Omega} \\ &\Omega = (a_{11}a_{22} - a_{12}^2)a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2\end{aligned} \tag{2.35}$$

The stresses $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$, $\tau_{\alpha\beta}$, $\tau_{\alpha\gamma}$, $\tau_{\beta\gamma}$ produce internal forces (T_1, T_2, S_1, S_2, N_1, N_2) and moments (M_1, M_2, H), which must satisfy the following satitical conditions [1, 2, 5]:

$$
\begin{gathered}
\frac{\partial}{\partial\alpha}(BT_1) - T_2\frac{\partial B}{\partial\alpha} + \frac{\partial}{\partial\beta}(AS_2) + S_1\frac{\partial A}{\partial\beta} + ABk_1N_1 = -ABX^*\\
\frac{\partial}{\partial\beta}(AT_2) - T_1\frac{\partial A}{\partial\beta} + \frac{\partial}{\partial\alpha}(BS_1) + S_2\frac{\partial B}{\partial\alpha} + ABk_2N_2 = -ABY^*\\
-(k_1T_1 + k_2T_2) + \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}(BN_1) + \frac{\partial}{\partial\beta}(AN_2)\right] = -Z^* \qquad (2.36)\\
\frac{\partial}{\partial\alpha}(BH) + H\frac{\partial B}{\partial\alpha} + \frac{\partial}{\partial\beta}(AM_2) - M_1\frac{\partial A}{\partial\beta} - ABN_2 = 0\\
\frac{\partial}{\partial\beta}(AH) + H\frac{\partial A}{\partial\beta} + \frac{\partial}{\partial\alpha}(BM_1) - M_2\frac{\partial\beta}{\partial\alpha} - ABN_1 = 0\\
S_1 - S_2 + k_1H - k_2H = 0
\end{gathered}
$$

In these formulas the symbols

$$X^* = X^*(\alpha,\ \beta), \qquad Y^* = Y^*(\alpha,\ \beta), \qquad Z^* = Z^*(\alpha,\ \beta)$$

represent the components of the intensity vector of the applied surface load, referred to the middle surface of the shell [7], namely

$$P^* = P^+\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) + P^-\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right) \qquad (2.37)$$

where P stands generally for X, Y, Z.

The stress resultants appearing in (2.36) are determined in the usual manner [1, 2, 10]. Without going into details, we give here the simplest elasticity formulas, which identically satisfy the sixth equation of statics:

$$
\begin{aligned}
T_1 = C_{11}\left(\varepsilon_1 + \frac{h^2}{12}\eta_1 + \frac{h^4}{80}\xi_1\right) + C_{12}\left(\varepsilon_2 + \frac{h^2}{12}\eta_2 + \frac{h^4}{80}\xi_2\right) +\\
+ C_{16}\left(\omega + \frac{h^2}{12}\lambda + \frac{h^4}{80}\zeta\right) \qquad (2.38)
\end{aligned}
$$

$$
\begin{aligned}
T_2 = C_{22}\left(\varepsilon_2 + \frac{h^2}{12}\eta_2 + \frac{h^4}{80}\xi_2\right) + C_{12}\left(\varepsilon_1 + \frac{h^2}{12}\eta_1 + \frac{h^4}{80}\xi_1\right) +\\
+ C_{26}\left(\omega + \frac{h^2}{12}\lambda + \frac{h^4}{80}\zeta\right) \qquad (2.39)
\end{aligned}
$$

$$
\begin{aligned}
S_1 = C_{16}\left(\varepsilon_1 + \frac{h^2}{12}\eta_1 + \frac{h^4}{80}\xi_1\right) + C_{26}\left(\varepsilon_2 + \frac{h^2}{12}\eta_2 + \frac{h^4}{80}\xi_2\right) +\\
+ C_{66}\left(\omega + \frac{h^2}{12}\lambda + \frac{h^4}{80}\zeta\right) + k_2\left[C_{16}\left(\frac{h^2}{12}\varkappa_1 + \frac{h^4}{80}\theta_1\right) + C_{26}\left(\frac{h^2}{12}\varkappa_2 + \frac{h^4}{80}\theta_2\right) +\right.\\
\left. + C_{66}\left(\frac{h^2}{12}\tau + \frac{h^4}{80}\lambda\right)\right] \qquad (2.40)
\end{aligned}
$$

$$S_2 = C_{26}\left(\varepsilon_2 + \frac{h^2}{12}\eta_2 + \frac{h^4}{80}\xi_2\right) + C_{16}\left(\varepsilon_1 + \frac{h^2}{12}\eta_1 + \frac{h^4}{80}\xi_1\right) +$$
$$+ C_{66}\left(\omega + \frac{h^2}{12}\lambda + \frac{h^4}{80}\zeta\right) + k_1\left[C_{26}\left(\frac{h^2}{12}\varkappa_2 + \frac{h^4}{80}\theta_2\right) + C_{16}\left(\frac{h^2}{12}\varkappa_1 + \frac{h^4}{80}\theta_1\right) + \right.$$
$$\left. + C_{66}\left(\frac{h^2}{12}\tau + \frac{h^4}{80}\lambda\right)\right] \quad (2.41)$$

$$M_1 = D_{11}\left(\varkappa_1 + \frac{3h^2}{20}\theta_1\right) + D_{12}\left(\varkappa_2 + \frac{3h^2}{20}\theta_2\right) + D_{16}\left(\tau + \frac{3h^2}{20}\lambda\right) \quad (2.42)$$

$$M_2 = D_{22}\left(\varkappa_2 + \frac{3h^2}{20}\theta_2\right) + D_{12}\left(\varkappa_1 + \frac{3h^2}{20}\theta_1\right) + D_{26}\left(\tau + \frac{3h^2}{20}\lambda\right) \quad (2.43)$$

$$H_1 = H_2 = H = D_{16}\left(\varkappa_1 + \frac{3h^2}{20}\theta_1\right) + D_{26}\left(\varkappa_2 + \frac{3h^2}{20}\theta_2\right) +$$
$$+ D_{66}\left(\tau + \frac{3h^2}{20}\lambda\right) \quad (2.44)$$

$$N_1 = \frac{h}{2}(X^+ - X^-) - \frac{h^3}{12}\varphi(\alpha, \beta) \quad (2.45)$$

$$N_2 = \frac{h}{2}(Y^+ - Y^-) - \frac{h^3}{14}\psi(\alpha, \beta) \quad (2.46)$$

In these relations we have the following formulas for the rigidity constants C_{ik} of compression and D_{ik} of bending:

$$C_{ik} = hB_{ik}, \quad D_{ik} = \frac{h^3}{12}B_{ik} \quad (2.47)$$

We state here that, in the process of substitution of the values of $\epsilon_1, \ldots, \zeta$, all terms containing X and Y can be omitted, still maintaining a sufficiently high degree of accuracy [7], in all elasticity relations.

Using the formulas (2.29), (2.30), we can eliminate the displacement components u, v, w of the middle surface from the relations (2.14) to (2.16); this leads to

$$k_2\varkappa_1^\circ + k_1\varkappa_2^\circ + \frac{1}{AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[B\frac{\partial\varepsilon_2}{\partial\alpha} + \frac{\partial B}{\partial\alpha}(\varepsilon_2 - \varepsilon_1) - \frac{A}{2}\frac{\partial\omega}{\partial\beta} - \frac{\partial A}{\partial\beta}\omega\right]\right\} +$$
$$+ \frac{1}{AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[A\frac{\partial\varepsilon_1}{\partial\beta} + \frac{\partial A}{\partial\beta}(\varepsilon_1 - \varepsilon_2) - \frac{B}{2}\frac{\partial\omega}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\omega\right]\right\} = 0 \quad (2.48)$$

The equation (2.48) is the third continuity relation for the deformation of the middle surface of the shell. As it should be expected, the relation does not differ in any way from the corresponding relation of the classical theory of thin shells [2, 5]. The remaining two conditions of continuity for the deformation of the middle surface will not be needed in the present paper.

The equations (2.14) to (2.31), (2.36), (2.38) to (2.46) taken together represent a complete system of equations of the theory of shells.

It is known [1,2,5] that such a complete system can be established in various ways. In view of its extreme complexity in the general case of a shell of arbitrary form, the complete system of equations will be considered here for one practically important type of shell only.

In the process of solving actual boundary value problems the differential equations of the shell have to be completed in the usual manner by statement of the boundary conditions [1,2, 3] .

3. Avoiding discussion of details, we mention here some possible special types of boundary conditions.

Free edge. This designation will characterize such an edge (α = const.) of the shell, for which

$$M_1 = 0, \quad H = 0, \quad S_1 = 0, \quad T_1 = 0, \quad N_1 = 0 \tag{3.1}$$

Simply supported edge. This designation will be used for such an edge (α = const.) of the shell, for which

$$M_1 = 0, \quad T_1 = 0, \quad w = 0, \quad v = 0, \quad B_{11}\theta_1 + B_{12}\theta_2 + B_{16}\lambda = 0 \tag{3.2}$$

Fixed edge with a hinge. This designation characterizes such an edge (α = const.) of the shell, for which

$$M_1 = 0, \quad u = 0, \quad v = 0, \quad w = 0, \quad \psi = 0 \tag{3.3}$$

Clamped edge. This designation refers to such an edge (α = const.) of the shell, for which

$$u = 0, \quad v = 0, \quad w = 0, \quad \psi = 0$$
$$\frac{1}{A}\frac{\partial w}{\partial \alpha} - k_1 u + \frac{h^2}{8}\Phi_1 = 0 \tag{3.4}$$

Of course, other boundary conditions are still possible.

The boundary conditions for an edge β = const. can be stated in an analogous manner.

Concluding this Section we note that the subject of the boundary conditions requires special investigations.

A detailed study of the results presented in the first three Sections of this paper reveals the following fact: the special case, characterized by $a_{44} = 0$, $a_{55} = 0$, $a_{45} = 0$, leads to the basic relations and equations of the theory of anisotropic shells based upon the hypotheses of nondeformable normals.

4. Consider a shell in the form of a circular cylinder of radius R. We take the α and β coordinate lines to be directed along the generators and the parallel circles of the middle surface, respectively. Assume that the shell is being acted upon by normally applied loading only. For such a shell

$$A = \text{const}, \qquad B = \text{const}, \qquad k_1 = 0, \qquad k_2 = \frac{1}{R} \tag{4.1}$$

The coefficients of the expansions (2.13) are

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial \alpha}, \qquad \varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{R}w, \qquad \omega = \frac{1}{B}\frac{\partial u}{\partial \beta} + \frac{1}{A}\frac{\partial v}{\partial \alpha} \tag{4.2}$$

$$\begin{aligned} \varkappa_1 &= -\frac{1}{A^2}\frac{\partial^2 w}{\partial \alpha^2} - \frac{h^2}{8}\frac{1}{A}\frac{\partial \Phi_1}{\partial \alpha} \\ \varkappa_2 &= -\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + \frac{1}{R}\frac{1}{B}\frac{\partial v}{\partial \beta} - \frac{h^2}{8}\frac{1}{B}\frac{\partial \Phi_2}{\partial \beta} \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tau &= -\frac{2}{AB}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{2}{R}\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{h^2}{8}\left(\frac{1}{B}\frac{\partial \Phi_1}{\partial \beta} + \frac{1}{A}\frac{\partial \Phi_2}{\partial \alpha}\right) \\ \eta_1 &= 0, \qquad \eta_2 = \frac{1}{R}\frac{1}{B^2}\frac{\partial^2 w}{\partial \beta^2} + \frac{h^2}{16}\frac{1}{R}\frac{1}{B}\frac{\partial \Phi_2}{\partial \beta} \\ \nu &= \frac{1}{R}\frac{1}{AB}\frac{\partial^2 w}{\partial \alpha \partial \beta} + \frac{h^2}{16}\frac{1}{R}\left(2\frac{1}{B}\frac{\partial \Phi_1}{\partial \beta} - \frac{1}{A}\frac{\partial \Phi_2}{\partial \alpha}\right) \end{aligned} \tag{4.4}$$

$$\theta_1 = \frac{1}{6A}\frac{\partial \Phi_1}{\partial \alpha}, \qquad \theta_2 = \frac{1}{6B}\frac{\partial \Phi_2}{\partial \beta}, \qquad \lambda = \frac{1}{6B}\frac{\partial \Phi_1}{\partial \beta} + \frac{1}{6A}\frac{\partial \Phi_2}{\partial \alpha} \tag{4.5}$$

$$\xi_1 = 0, \qquad \xi_2 = -\frac{1}{8B}\frac{1}{R}\frac{\partial \Phi_2}{\partial \beta}, \qquad \zeta = -\frac{1}{6R}\left(\frac{1}{B}\frac{\partial \Phi_1}{\partial \beta} + \frac{1}{4A}\frac{\partial \Phi_2}{\partial \alpha}\right) \tag{4.6}$$

The equations of equilibrium assume the form

$$\begin{aligned} &\frac{1}{A}\frac{\partial T_1}{\partial \alpha} + \frac{1}{B}\frac{\partial S_2}{\partial \beta} = 0, \qquad && \frac{1}{A}\frac{\partial H}{\partial \alpha} + \frac{1}{B}\frac{\partial M_2}{\partial \beta} - N_2 = 0 \\ &\frac{1}{B}\frac{\partial T_2}{\partial \beta} + \frac{1}{A}\frac{\partial S_1}{\partial \alpha} + \frac{1}{R}N_2 = 0, \qquad && \frac{1}{B}\frac{\partial H}{\partial \beta} + \frac{1}{A}\frac{\partial M_1}{\partial \alpha} - N_1 = 0 \\ &\frac{1}{A}\frac{\partial N_1}{\partial \alpha} + \frac{1}{B}\frac{\partial N_2}{\partial \beta} - \frac{1}{R}T_2 = -Z \end{aligned} \tag{4.7}$$

Substituting the expressions for ϵ_1, ..., ζ from (4.2) to (4.6) into the formulas (2.38) to (2.46), we obtain the stress resultants in terms of the unknown functions u, v, w, ϕ, ψ. Substituting the obtained expressions of the stress resultants into the equations of equilibrium (4.7), we find a final system of the five differential equations for the five unknown functions u, v, w, ϕ, ψ, namely

$$\begin{aligned} &\nabla_1(C_{ik})u + \nabla_6(C_{ik})v + \Big\{C_{12}\frac{1}{A}\frac{\partial}{\partial \alpha} + C_{26}\frac{1}{B}\frac{\partial}{\partial \beta} + \\ &+ \frac{h^2}{12}\left[(C_{12} + C_{66})\frac{1}{AB^2}\frac{\partial^3}{\partial \alpha \partial \beta^2} + C_{16}\frac{1}{A^2 B}\frac{\partial^3}{\partial \alpha^2 \partial \beta} + C_{26}\frac{1}{B^3}\frac{\partial^3}{\partial \beta^3}\right]\Big\}\frac{w}{R} + \\ &+ Q_4(C_{ik}, a_{ik})\psi + Q_5(C_{ik}, a_{ik})\varphi = 0 \end{aligned} \tag{4.8}$$

$$\nabla_6 (C_{ik}) u + \nabla_2 (C_{ik}) v + \Big\{ C_{22} \frac{1}{B} \frac{\partial}{\partial \beta} + C_{26} \frac{1}{A} \frac{\partial}{\partial \alpha} +$$

$$+ \frac{h^2}{12} \left[C_{22} \frac{1}{B^3} \frac{\partial^3}{\partial \beta^3} - C_{16} \frac{1}{A^3} \frac{\partial^3}{\partial \alpha^3} + C_{26} \frac{1}{AB^2} \frac{\partial^3}{\partial \alpha \partial \beta^2} - C_{66} \frac{1}{A^2 B} \frac{\partial^3}{\partial \alpha^2 \partial \beta} \right] \Big\} \frac{w}{R} +$$

$$+ R_4 (C_{ik}, a_{ik}) \psi + R_5 (C_{ik}, a_{ik}) \varphi = 0 \tag{4.9}$$

$$\left(C_{12} \frac{1}{A} \frac{\partial}{\partial \alpha} + C_{26} \frac{1}{B} \frac{\partial}{\partial \beta} \right) \frac{u}{R} + \left(C_{22} \frac{1}{B} \frac{\partial}{\partial \beta} + C_{26} \frac{1}{A} \frac{\partial}{\partial \alpha} \right) \frac{v}{R} +$$

$$+ \left[C_{22} + \frac{h^2}{12} \left(C_{22} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + C_{26} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \, \partial \beta} \right) \right] \frac{w}{R^2} +$$

$$+ P_4 (C_{ik}, a_{ik}) \psi + P_5 (C_{ik}, a_{ik}) \varphi = Z \tag{4.10}$$

$$\left(D_{22} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + 2 D_{66} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + 3 D_{26} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \, \partial \beta} \right) \frac{v}{R} -$$

$$- E_2 (D_{ik}) w - S_4 (D_{ik}, a_{ik}) \psi - S_5 (D_{ik}, a_{ik}) \varphi = 0 \tag{4.11}$$

$$\left[D_{26} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + 2 D_{16} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + (2 D_{66} + D_{12}) \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta} \right] \frac{v}{R} -$$

$$- E_1 (D_{ik}) w - K_4 (D_{ik}, a_{ik}) \psi - K_5 (D_{ik}, a_{ik}) \varphi = 0 \tag{4.12}$$

where

$$\nabla_1 (C_{ik}) = C_{11} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + C_{66} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + 2 C_{16} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta}$$

$$\nabla_2 (C_{ik}) = C_{22} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} + C_{66} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + 2 C_{26} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta}$$

$$\nabla_6 (C_{ik}) = C_{16} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + (C_{12} + C_{66}) \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta} + C_{26} \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2}$$

$$E_1 (D_{ik}) = D_{11} \frac{1}{A^3} \frac{\partial^3}{\partial \alpha^3} + 3 D_{16} \frac{1}{A^2 B} \frac{\partial^3}{\partial \alpha^2 \partial \beta} +$$

$$+ (D_{12} + 2 D_{66}) \frac{1}{AB^2} \frac{\partial^3}{\partial \alpha \partial \beta^2} + D_{26} \frac{1}{B^3} \frac{\partial^3}{\partial \beta^3}$$

$$E_2 (D_{ik}) = D_{22} \frac{1}{B^3} \frac{\partial^3}{\partial \beta^3} + 3 D_{26} \frac{1}{AB^2} \frac{\partial^3}{\partial \beta^2 \partial \alpha} +$$

$$+ (D_{12} + 2 D_{66}) \frac{1}{BA^2} \frac{\partial^3}{\partial \beta \partial \alpha^2} + D_{16} \frac{1}{A^3} \frac{\partial^3}{\partial \alpha^3}$$

$$P_i (C_{ik}, a_{ik}) = \left[(i - 4) \frac{h^3}{12} - C_{26} \frac{3 h^4}{640 R^2} a_{4i} \right] \frac{1}{A} \frac{\partial}{\partial \alpha} +$$

$$+ \left[\frac{h^4}{120 R^2} \left(\frac{7}{16} C_{22} a_{4i} + C_{26} a_{i5} \right) - (i - 5) \frac{h^3}{12} \right] \frac{1}{B} \frac{\partial}{\partial \beta}$$

$$R_i (C_{ik}, a_{ik}) = (i - 5) \frac{h^3}{12 R} + \frac{h^4}{120 R} \left[\left(\frac{7}{16} C_{22} a_{4i} + C_{26} a_{i5} \right) \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} - \right.$$

$$\left. - \frac{9}{8} C_{26} a_{4i} \frac{1}{AB} \frac{\partial^2}{\partial \alpha \partial \beta} - \left(\frac{25}{16} C_{66} a_{4i} + C_{16} a_{i5} \right) \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} \right]$$

$$Q_i (C_{ik}, a_{ik}) = \frac{h^4}{120 R} \left[\left(\frac{7}{16} C_{12} a_{4i} - \frac{9}{16} C_{66} a_{4i} + C_{16} a_{i5} \right) \frac{\partial^2}{\partial \alpha \partial \beta} - \right.$$

$$\left. - \frac{9}{16} C_{16} a_{4i} \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + \left(\frac{7}{16} C_{26} a_{4i} + C_{66} a_{i5} \right) \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} \right]$$

$$S_i(D_{ik}, a_{ik}) = \frac{h^2}{10}\Big[(D_{16}a_{i5} + D_{66}a_{4i})\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (2D_{26}a_{4i} + D_{66}a_{i5} + D_{12}a_{i5})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + (D_{22}a_{4i} + D_{26}a_{i5})\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big] - (i-5)\frac{h^3}{12}$$

$$K_i(D_{ik}, a_{ik}) = \frac{h^2}{10}\Big[(D_{11}a_{i5} + D_{16}a_{4i})\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + (2D_{16}a_{i5} + D_{66}a_{4i} + D_{12}a_{4i})\frac{1}{AB}\frac{\partial^2}{\partial\alpha\partial\beta} + (D_{26}a_{4i} + D_{66}a_{i5})\frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}\Big] + (i-4)\frac{h^3}{12}$$

Thus, the problem of the anisotropic cylindrical shell is reduced to a system of five differential equations (4.8) to (4.12) for the five unknown functions. Having obtained the latter, we will find without difficulty the stress resultants, as well as the stresses, by means of the formulas (2.32) to (2.34), (2.38) to (2.46) and (4.2) to (4.6).

The system of equations (4.8) to (4.12) undergoes substantial simplification in the case of a transversely isotropic shell [10]. It is known that for a transversely isotropic solid we have

$$a_{16} = 0, \quad a_{26} = 0, \quad a_{45} = a_{54} = 0, \quad a_{44} = a_{55} = \frac{1}{G'}$$
$$B_{11} = B_{22} = \frac{E}{1-\mu^2}, \quad B_{12} = \mu B_{11}, \quad B_{66} = \frac{E}{2(1+\mu)} \tag{4.13}$$

where E is the modulus of elasticity in the plane of isotropy, μ is Poisson's ratio, G' is the shear modulus for planes normal to the plane of isotropy.

We assume the plane of isotropy of the material to be parallel, at each point of the shell, to the middle surface of the latter.

The coordinates α, β are to be chosen in such a manner that the coefficients of the first quadratic form assume the following values [1,2]:

$$A = 1, \quad B = R \tag{4.14}$$

By virtue of (4.13) and (4.14) the final system of equations becomes simpler and assumes the following form:

$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{1-\mu}{2R^2}\frac{\partial^2 u}{\partial\beta^2} + \frac{1+\mu}{2R}\frac{\partial^2 v}{\partial\alpha\,\partial\beta} + \frac{\mu}{R}\frac{\partial w}{\partial\alpha} + \frac{(1+\mu)h^2}{24R^3}\frac{\partial^3 w}{\partial\alpha\partial\beta^2} + \frac{23\mu-9}{3840}\frac{h^4}{R^2}a_{44}\frac{\partial^2\psi}{\partial\alpha\,\partial\beta} + \frac{1-\mu}{240}\frac{h^4}{R^3}a_{44}\frac{\partial^2\varphi}{\partial\beta^2} = 0 \tag{4.15}$$

$$\frac{1+\mu}{2R}\frac{\partial^2 u}{\partial\alpha\partial\beta} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial\alpha^2} + \frac{1}{R^2}\frac{\partial^2 v}{\partial\beta^2} + \frac{1}{R^2}\frac{\partial w}{\partial\beta} + \frac{h^2}{12R^4}\frac{\partial^3 w}{\partial\beta^3} - \frac{(1-\mu)h^2}{24R^3}\frac{\partial^3 w}{\partial\alpha^2\partial\beta} + \frac{7h^4}{1920R^3}a_{44}\frac{\partial^2\psi}{\partial\beta^2} - \frac{5(1-\mu)h^4}{768R}a_{44}\frac{\partial^2\psi}{\partial\alpha^2} - \frac{(1-\mu^2)h^2}{12ER}\psi = 0 \tag{4.16}$$

$$\frac{\mu}{R}\frac{\partial u}{\partial \alpha}+\frac{1}{R^2}\frac{\partial v}{\partial \beta}+\frac{w}{R^2}+\frac{h^2}{12R^4}\frac{\partial^2 w}{\partial \beta^2}+\frac{7h^4}{1920R^3}a_{44}\frac{\partial \psi}{\partial \beta}+ \\ +\frac{(1-\mu^2)h^2}{12ER}\frac{\partial \psi}{\partial \beta}+\frac{(1-\mu^2)h^2}{12E}\frac{\partial \varphi}{\partial \alpha}=\frac{1-\mu^2}{Eh}Z \tag{4.17}$$

$$\frac{1-\mu}{R}\frac{\partial^2 v}{\partial \alpha^2}+\frac{1}{R^3}\frac{\partial^2 v}{\partial \beta^2}-\frac{1}{R}\frac{\partial^3 w}{\partial \alpha^2 \partial \beta}-\frac{1}{R^3}\frac{\partial^3 w}{\partial \beta^3}+\frac{1-\mu^2}{E}\psi- \\ -\frac{h^2}{10}a_{44}\left(\frac{1-\mu}{2}\frac{\partial^2 \psi}{\partial \alpha^2}+\frac{1}{R^2}\frac{\partial^2 \psi}{\partial \beta^2}+\frac{1+\mu}{2R}\frac{\partial^2 \varphi}{\partial \alpha\,\partial \beta}\right)=0 \tag{4.18}$$

$$\frac{1}{R^2}\frac{\partial^2 v}{\partial \alpha \partial \beta}-\frac{1}{R^2}\frac{\partial^3 w}{\partial \alpha \partial \beta^2}-\frac{\partial^3 w}{\partial \beta^3}- \\ -\frac{h^2}{10}a_{44}\left(\frac{1+\mu}{2R}\frac{\partial^2 \psi}{\partial \alpha \partial \beta}+\frac{\partial^2 \varphi}{\partial \alpha^2}+\frac{1-\mu}{2}\frac{1}{R^2}\frac{\partial^2 \varphi}{\partial \beta^2}\right)+\frac{1-\mu^2}{E}\varphi=0 \tag{4.19}$$

As an example we shall treat here the problem of a horizontal tube, of transversely isotropic material, simply supported at its ends. The tube is entirely filled with a liquid of specific weight γ. The weight of the tube material shall be neglected [7, 12].

Measuring the angle β from the lowest point of the cross section of the tube, we use the expansions

$$u=\sum_m\sum_n A_{mn}\cos n\beta\cos\frac{m\pi\alpha}{l},\qquad \varphi=\sum_m\sum_n D_{mn}\cos n\beta\cos\frac{m\pi\alpha}{l}$$
$$v=\sum_m\sum_n B_{mn}\sin n\beta\sin\frac{m\pi\alpha}{l},\qquad \psi=\sum_m\sum_n E_{mn}\sin n\beta\sin\frac{m\pi\alpha}{l} \tag{4.20}$$
$$w=\sum_m\sum_n C_{mn}\cos n\beta\sin\frac{m\pi\alpha}{l},$$

The chosen functions fulfil the boundary conditions of simple support along the edges $\alpha = 0$, $\alpha = l$, as well as the conditions of periodicity with the period 2π for the argument β. The acting load, the radial pressure of the fluid, is

$$q=R\gamma(1+\cos\beta) \tag{4.21}$$

It can be represented by the double series

$$Z=\sum_m\sum_n q_{mn}\cos n\beta\sin\frac{m\pi\alpha}{l} \tag{4.22}$$

where the coefficients q_{mn} are given [7, 12] by

$$q_{mn}=0,\quad q_{m0}=\frac{4\gamma R}{m\pi},\quad q_{m1}=\frac{4\gamma R}{m\pi} \tag{4.23}$$

In view of the good convergence of the expansions with respect to the subscript m = 1, 3, 5, ... we will confine ourselves in the following to the first term.

Substituting the functions u, v, w, ϕ, ψ from (4.20), and the function

z from (4.22) into the corresponding equations of the system (4.15) to (4.19), we obtain, for each pair of values of m and n, a system of five equations for the five unknown coefficients A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}. In the special case, when $n = 0$, these systems undergo essential simplifications.

Let us consider the numerical example treated in [7,12]; take $a = 50$ cm, $l = 25$ cm, $h = 7$ cm, while $\mu = 0.3$. For the dimensions just given we shall examine three cases, for which the ratio E/G' equals 2.6; 5.0; 10.0, respectively.

In the case $E/G' = 2.6$ we have evidently to deal with an isotropic shell, while in the second and in the third case we have transversely isotropic shells.

The values of the coefficients C_{mn} of the normal displacement component of the shell are given in Table 1 in the form of the ratio C_{mn}/N, where $N = 24\gamma R^3 l^2/E\pi h$. In the last column of Table 1 are given the values of

TABLE 1.

$\frac{E}{G'}$	$\frac{10^4}{N} C_{01}$	$\frac{10^4}{N} C_{11}$	$\frac{10^4}{N} C$
	0.7022	0.6708	1.3730
2.6	0.8103	0.8004	1.6107
5.0	0.9000	0.9138	1.8138
10.0	1.0616	1.0275	2.0891

the coefficient of the maximum normal displacement, i.e. the values of the coefficient of w at the point $\beta = 0$, $\beta = \frac{1}{2} l$.

For comparison we give in the first line of Table 1 the values of the same coefficients C_{mn}/N, where $N = 24\gamma R^3 l^2/E\pi h$, calculated by means of the theory based upon the hypothesis of non-deformable normals [7,12].

The comparison shows that the results obtained on the basis of the latter theory essentially differ from those derived from the theory offered in the present paper. We see that even in the case of an isotropic shell the error incurred in the classical theory (based upon the hypothesis of non-deformable normals) can amount to 15%. In the case of transversely isotropic shells the error can become quite substantial for the case of the example considered here, depending on the ratio E/G'. For instance, in the case of a ratio $E/G' = 10$ the error just mentioned rises to 35%.

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